CS 374B Review MT3

Turing Complete Spookiness

ACM @ UIUC

November 3, 2024

Disclaimers and Logistics

- **Disclaimer:** Some of us are CAs, but we have not seen the exam. We have no idea what the questions are. However, we've taken the course and reviewed Kani's previous exams, so we have **suspicions** as to what the questions will be like.
- This review session is being recorded. Recordings and slides will be distributed on EdStem after the end.
- **Agenda:** We'll quickly review all topics likely to be covered, then go through a practice exam, then review individual topics by request.
  - Questions are designed to be written in the same style as Kani's previous exams but to be *slightly* harder, so don't worry if you don't get everything right away!
- Please let us know if we're going too fast/slow, not speaking loud enough/speaking too loud, etc.
- If you have a question anytime during the review session, please ask! Someone else almost surely has a similar question.
- We'll provide a feedback form at the end of the session.

P and NP

- A **decision problem** is a problem with a true/false answer. (yes/no, etc.)
- **P** is the set of decision problems with a polynomial-time solver.
- **NP** is the set of decision problems with a polynomial-time *nondeterministic* solver.
- Alternatively, NP is the set of decision problems with a polynomial-time *certifier* for "true" answers, given a polynomial-size *certificate*.
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Formally, an algorithm C is a certifier for problem X when $s \in X$ if and only if there exists string t such that $C(s, t) =$ true.

- t here is a "certificate."
- We can show X is NP by providing this information, and showing C is polynomial-time and t is polynomial-size (with respect to the size of the input s).

co-NP

- **co-NP** is the set of decision problems X whose complements $\overline{X}$ are in NP.
- Alternatively, NP is the set of decision problems with a polynomial-time certifier for **"false"** answers, given a polynomial-size certificate.
- For example, the problem of deciding whether a graph *doesn't* have a Hamiltonian path is in co-NP.

co-NP isn't on your skillset, but be aware that this is *not* the same thing as NP.

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Make sure you're going in the right direction!

If you're trying to prove that a problem is NP-hard or undecidable, you need to reduce **from** an NP-hard/undecidable problem **to** the problem you want to prove is hard (in other words, show that an oracle for your problem can be used to solve an NP-hard/undecidable problem). The most common mistake on exams is reducing in the wrong direction.

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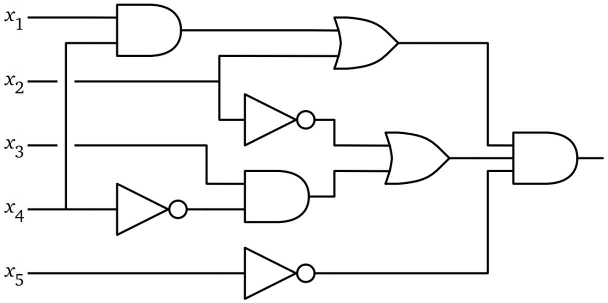
Template- Reduction

Assume that there exists an oracle function *B* which runs in [TIME CONSTRAINT]. Thus, we can solve *A* as follows:

```
procedure A(input):
    Do some preprocessing to create instances of problem B
    outputs ← B(generated inputs)
    Do some postprocessing on outputs to get the correct answer for A
```

A Tour of NP-Hard Problems: CircuitSAT and 3SAT

- **CircuitSAT**: The “original” NP-complete problem. Given a boolean circuit, is there a set of inputs that makes it return `true`?



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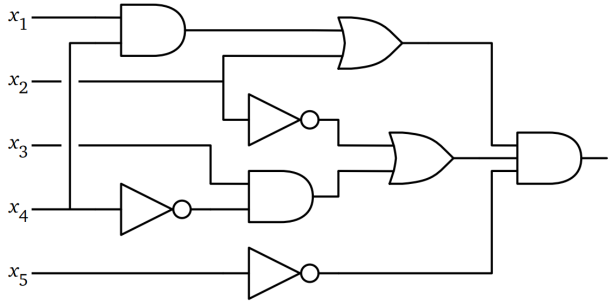

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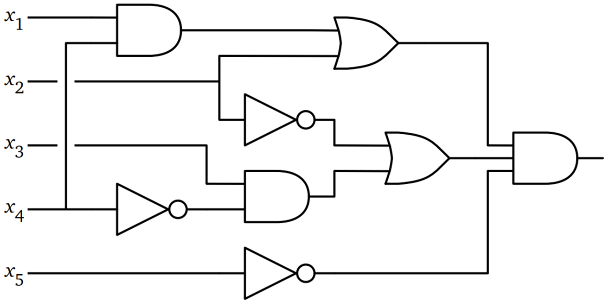

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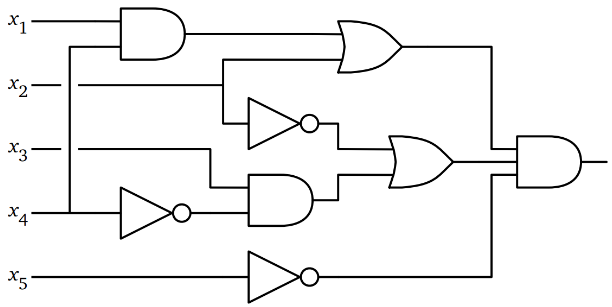

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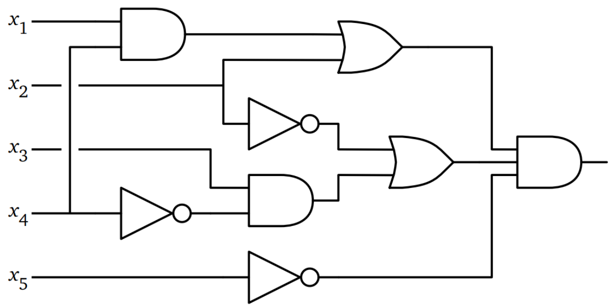

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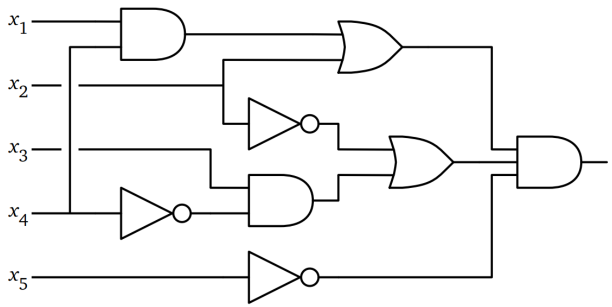

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Be careful with *k*-SAT variants!

While k-SAT for $k \geq 3$ is NP-complete, there is a polynomial-time algorithm for 2SAT. (Using strongly connected components!)

A Tour of NP-Hard Problems: CircuitSAT and 3SAT

Consider the problem **MajSAT**: Clauses now consist of 5 literals, and you must satisfy at least 3 literals in each clause. Is **MajSAT** in NP, NP-hard, both, or neither? Prove why by either stating an algorithm or providing a reduction.

A Tour of NP-Hard Problems: Max{Clique, IndSet}, MinVertexCover

- **MaxClique**: Given a graph G and positive integer h, can we find a K_h subgraph in G (i.e. a set of h nodes where each one has an edge to every other)?

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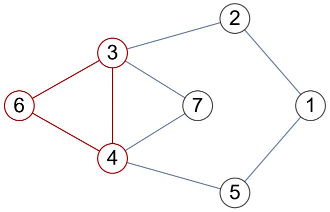

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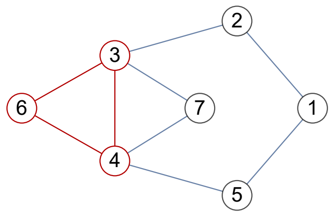

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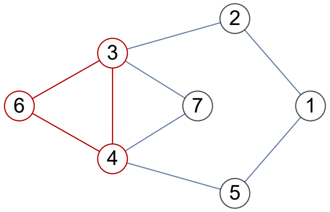

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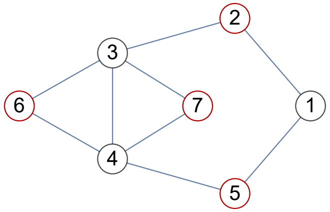

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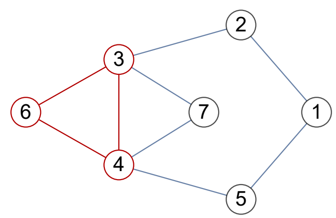

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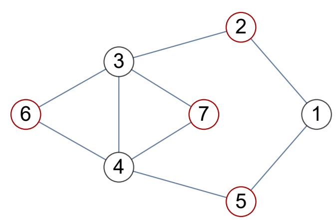

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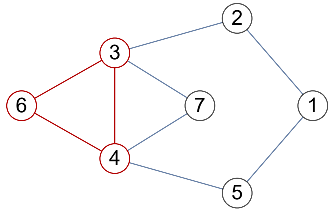

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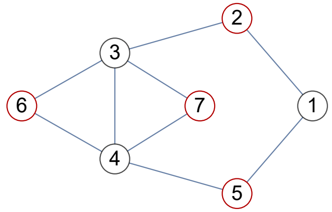

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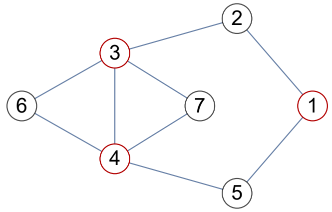
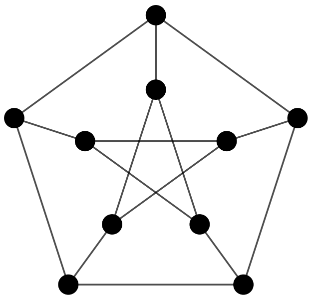


A Tour of NP-Hard Problems: Max{Clique, IndSet}, MinVertexCover

ACM is writing their review session for CS/ECE 374B MT3. While making slides, each CA writes 2 problems, either alone or in collaboration with other CAs. Since all of the CAs all have inflated egos, they won't show up to the review session unless one of the problems that they worked on is in the review session. Show that determining whether we can run a review session with at most k problems is NP-complete.

A Tour of NP-Hard Problems: Graph Coloring

- Given an (undirected) graph, can we color the nodes with at most *k* colors so that no two vertices that share an edge are of the same color?



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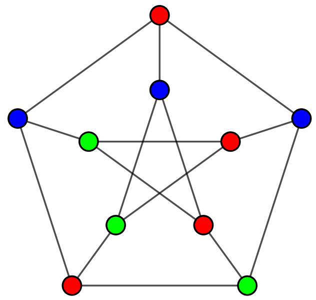

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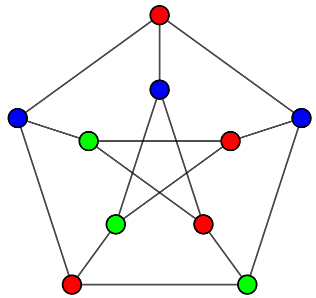

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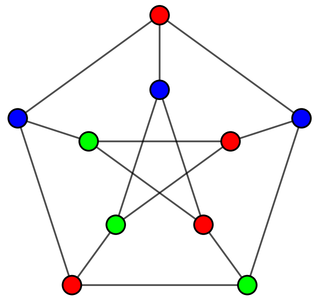

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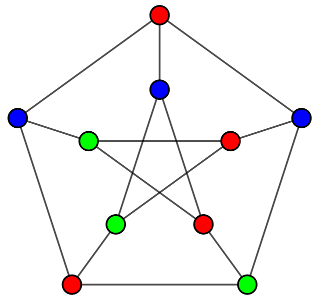

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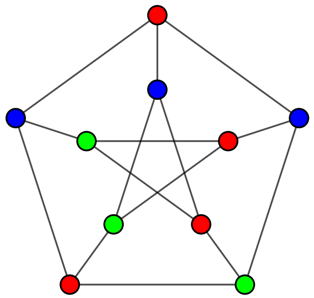

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Be careful with *k*-coloring variants!

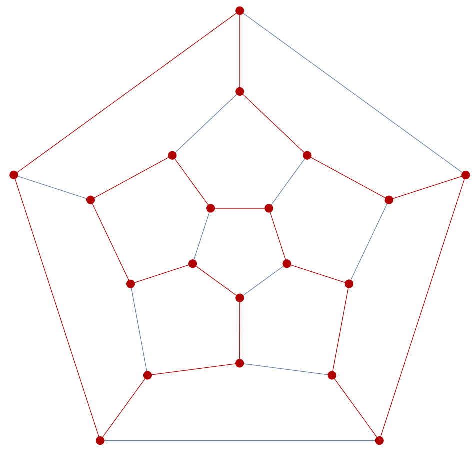
While k-coloring for $k \geq 3$ is NP-complete, you can find whether a graph is bipartite (2-colorable) using DFS.

A Tour of NP-Hard Problems: Graph Coloring

Consider the problem **Safe7Color**, which asks you to color a graph with 7 colors, such that it is a violation if there is an edge $u \leftrightarrow v$ where $c(u)$ and $c(v)$ differ by 0 or 1 (mod 7). Is this problem NP-hard?

A Tour of NP-Hard Problems: Hamiltonian Paths and Cycles

- A **Hamilton Path** is a path that goes through each vertex *exactly* once. Likewise, a **Hamiltonian Cycle** is a cycle that goes through each node *exactly* once.
  - Every graph with a Hamiltonian cycle has a Hamiltonian path, but not every graph with a Hamiltonian cycle has a Hamiltonian path.



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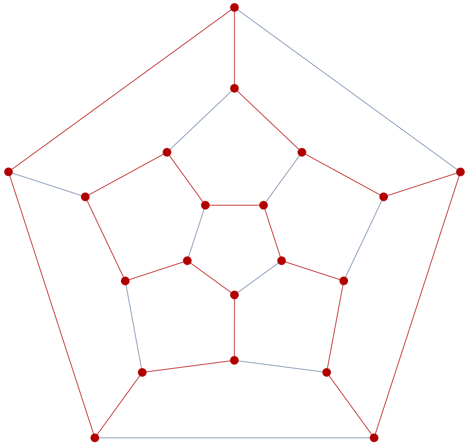

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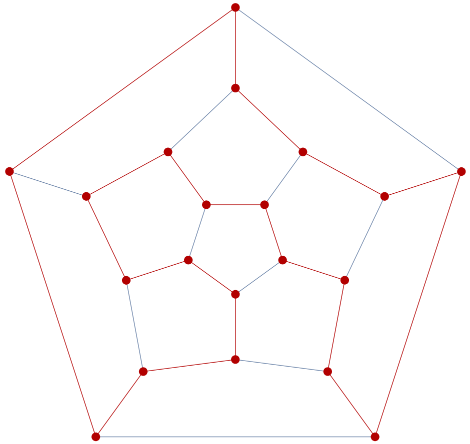
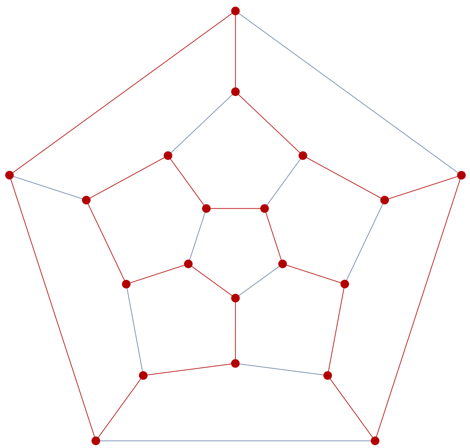


- Consider reducing from **HamPath** or **HamCycle** if. . .
  - You're given a graph, and you're asked to find a sequence of vertices

A Tour of NP-Hard Problems: Hamiltonian Paths and Cycles

- A **Hamilton Path** is a path that goes through each vertex *exactly* once. Likewise, a **Hamiltonian Cycle** is a cycle that goes through each node *exactly* once.
    - Every graph with a Hamiltonian cycle has a Hamiltonian path, but not every graph with a Hamiltonian cycle has a Hamiltonian path.



- Consider reducing from **HamPath** or **HamCycle** if. . .
    - You're given a graph, and you're asked to find a sequence of vertices
    - You have a resource pool, and you want to use up everuthing

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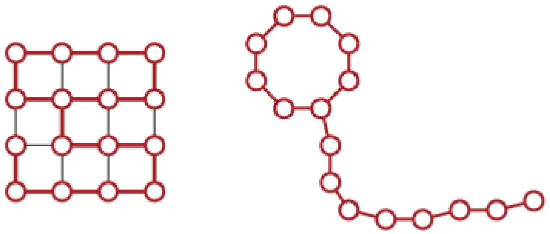
A **balloon graph** of size ℓ is a cycle of length ℓ attached to a path of length ℓ, where the cycle and the path are disjoint except for the connecting vertex. Show that it is NP-hard to determine whether a graph has a balloon subgraph of size at least k.



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- **Checkers**: given a $n \times n$ checkerboard, is there a move that captures at least k checkers?

Undecidability

- A language is **decidable** if there exists an algorithm which always returns `true` to all inputs in L and `false` to inputs not in L
  - If we can only return `true` to all inputs in L and either return `false` *or* infinite-loop for all other inputs, the language is merely **acceptable**.

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- *There is a Turing machine Y such that Accept(Y) $\in \mathcal{L}$.*
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Then, the language $AcceptIn(\mathcal{L}) \leftarrow \{\langle M \rangle \mid Accept(M) \in \mathcal{L}\}$ *is undecidable.*

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  3. Abuse the fact that you can put code into a function to derive a contradiction.

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(f) NeverLeft $= \{\langle M, w \rangle : M \text{ never moves left on input } w\}$

Feedback

- Please fill out the feedback form:
  `go.acm.illinois.edu/cs374b_mt3_feedback`